Inequality of George Apostolopuolos.

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Proposed by George Apostolopuolos.

Let a,b,c be positive real numbers with $a+b+c=3$. Prove that

$$\sqrt{\frac{ab}{2a+b+c}}+\sqrt{\frac{bc}{2b+c+a}}+\sqrt{\frac{ca}{2c+a+b}}\le\frac{3}{2}$$

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Since triples $\left(\sqrt{bc},\sqrt{ca},\sqrt{ab}\right),\left(\frac{1}{\sqrt{2a+b+c}},\frac{1}{\sqrt{2b+c+a}},\frac{1}{\sqrt{2c+a+b}}\right)$ agreed in order (because $\left(\sqrt{bc}-\sqrt{ca}\right)\left(\frac{1}{\sqrt{2a+b+c}}-\frac{1}{\sqrt{2b+c+a}}\right)\ge 0$) then by Rearrangement Inequality $\sum\sqrt{\frac{ab}{2a+b+c}}\le\sum\sqrt{\frac{bc}{2a+b+c}}$.

By replacing (x,y,z) in inequality $x+y+z\le\sqrt{3(x^2+y^2+z^2)}$ with $\left(\frac{bc}{\sqrt{2a+b+c}},\frac{ca}{\sqrt{2b+c+a}},\frac{ab}{\sqrt{2c+a+b}}\right)$ we obtain

$\sum\sqrt{\frac{bc}{2a+b+c}}\le\sqrt{3\sum\frac{bc}{2a+b+c}}$.

Thus, $\sum\sqrt{\frac{ab}{2a+b+c}}\le\sqrt{3\sum\frac{bc}{2a+b+c}}$ and remains to prove inequality

$\sqrt{3\sum\frac{bc}{2a+b+c}}\le\frac{3}{2}\Leftrightarrow\sum\frac{bc}{2a+b+c}\le\frac{3}{4}$ or, in homogeneous form,

(**1**) $\qquad\sum\frac{bc}{2a+b+c}\le\frac{a+b+c}{4.}$.

Using in (**1**) normalization $a+b+c=1$ we obtain inequality

$\sum\frac{bc}{2a+b+c}\le\frac{1}{4}\Leftrightarrow\sum\frac{bc}{a+1}\le\frac{1}{4}\Leftrightarrow 4\sum bc(b+1)(c+1)\le\prod(a+1)$.

We will prove that $4\sum bc(b+1)(c+1)\le\prod(a+1)$ holds for any $a,b,c\ge 0$.

Let $p:=ab+bc+ca,q:=abc$. Then $4\sum bc(b+1)(c+1)=$

$4\sum(b^2c^2+bc(b+c)+bc)=4\left(\left(\sum bc\right)^2-2abc\sum a+\sum a\cdot\sum bc-3abc+\sum bc\right)=$

$4(p^2-5q+2p),\prod(a+1)=2+p+q$ and $\prod(a+1)-4\sum bc(b+1)(c+1)=$

$2+p+q-4(p^2-5q+2p)=2-4p^2-7p+21q$.

Since $3p=3(ab+bc+ca)\le(a+b+c)^2=1)$ and $9q\ge 4p-1$

(Schure's Inequality $\sum a(a-b)(a-c)\ge 0$ in p,q notation and normalized by $a+b+c=1$) then $p\le 1/3$ and $q\ge\max\left\{\frac{4p-1}{9},0\right\}$.

If $p\in[1/4,1/3]$ then $2-4p^2-7p+21q\ge 2-\left(7p+4p^2-21\cdot\frac{4p-1}{9}\right)=$

$\frac{1}{3}(4p-1)(1-3p)\ge 0$;

If $p\in(0,1/4]$ then $2-4p^2-7p+21q\ge 2-7p-4p^2=(p+2)(1-4p)\ge 0$.